Online Appendix to "Dynamic Coordination with Payoff and Informational Externalities"

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OA.1 Omitted Proofs for Section 3

OA.1.1 Proof of Lemma 6

Leader x's expected payoff from stopping at t is

$$\mathcal{L}(x,t) = \lim_{\varepsilon\to 0}\left(q_L(x)\int_0^{t-\varepsilon} e^{-r\tau}\mathrm{d}G_F^1(\tau)H - (1-q_L(x))\int_0^{t-\varepsilon} e^{-r\tau}\mathrm{d}G_F^0(\tau)L\right).$$

Follower y's expected payoff from stopping at t is

$$\begin{aligned}\mathcal{F}(y,t) =& e^{-rt}\left(q_F(y)\left((1-G_L^1(t))H + (1-G_L^0(t))L\right) - (1-G_L^0(t))L\right)\\ &- e^{-rt}\lim_{\varepsilon\to 0}\left(q_F(y)(1-G_L^1(t-\varepsilon)) + (1-q_F(y))(1-G_L^0(t-\varepsilon))\right)c.\end{aligned}$$

I show the leader's expected payoff is supermodular and the follower's is submodular.

Denote $\Delta\mathcal{L}(x,t,t') = \mathcal{L}(x,t') - \mathcal{L}(x,t)$. For $t' > t$ and $x' > x$,

$$\begin{aligned}&\Delta\mathcal{L}(x',t,t') - \Delta\mathcal{L}(x,t,t')\\ =&\lim_{\varepsilon\to 0}(q_L(x') - q_L(x))\left(\int_{t-\varepsilon}^{t'-\varepsilon} e^{-r\tau}\mathrm{d}G_F^1(\tau)H + \int_{t-\varepsilon}^{t'-\varepsilon} e^{-r\tau}\mathrm{d}G_F^0(\tau)L\right).\end{aligned}$$

By MLRP, $q_L(x') - q_L(x) > 0$. For $t' > t$, $G_F^\theta(t') \geq G_F^\theta(t)$. So $\Delta\mathcal{L}(x',t,t') - \Delta\mathcal{L}(x,t,t') > 0$. Therefore, $\mathcal{L}(x,t)$ is supermodular in (x,t). By Topkis's theorem, $\sigma_L(x) = \arg\max_{t\geq 0}\mathcal{L}(x,t)$ is non-decreasing in x.

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Denote $\Delta\mathcal{F}(y,t,t') = \mathcal{F}(y,t') - \mathcal{F}(y,t)$. For $t' > t$ and $y' > y$,

$$
\begin{aligned}
&\Delta\mathcal{F}(y',t,t') - \Delta\mathcal{F}(y,t,t') \\
=&(q_F(y') - q_F(y))\left(e^{-rt'}(1-G_L^1(t')) - e^{-rt}(1-G_L^1(t))\right) H \\
&- (q_F(y') - q_F(y))\left(e^{-rt}(1-G_L^0(t)) - e^{-rt'}(1-G_L^0(t'))\right) L \\
&- \lim_{\varepsilon\to 0} c\Bigg(e^{-r(t'-\varepsilon)}(q_F(y') - q_F(y))\left((1-G_L^1(t'-\varepsilon)) + (1-G_L^0(t'-\varepsilon))\right) \\
&\qquad\qquad + e^{-r(t-\varepsilon)}(q_F(y') - q_F(y))\left((1-G_L^1(t-\varepsilon)) + (1-G_L^0(t-\varepsilon))\right)\Bigg).
\end{aligned}
$$

By MLRP, $q_F(y') - q_F(y) > 0$. For $t' > t$, $e^{-rt'}(1-G_L^\theta(t')) < e^{-rt}(1-G_L^\theta(t')) \leq e^{-rt}(1-G_L^\theta(t))$. So $\Delta\mathcal{F}(y',t,t') - \Delta\mathcal{F}(y,t,t') < 0$. Therefore, $\mathcal{F}(y,t)$ is submodular in (y,t). By Topkis's theorem, $\sigma_F(y) = \arg\max_{t\geq 0}\mathcal{F}(y,t)$ is non-increasing in y.

OA.2 Omitted Proofs for Section 4

OA.2.1 Proof of Lemma 10

Define $Q^\theta(\mu) := (1-F^\theta(\mu))/(1-\hat{F}^\theta(\mu))$. It follows directly from (7) that $h(\mu) > \hat{h}(\mu)$ if and only if $Q^1(\mu) < Q^0(\mu)$. Moreover, by (7), for all $\mu \in (0,1)$,

$$
\frac{f^0(\mu)}{\hat{f}^0(\mu)} = \frac{f^1(\mu)}{\hat{f}^1(\mu)} = \frac{f^0(\mu)+f^1(\mu)}{\hat{f}^0(\mu)+\hat{f}^1(\mu)}. \tag{OA.1}
$$

Because $F \succ_{\mathrm{ULR}} \hat{F}$, all three ratios in (OA.1) are unimodal and symmetric about $1/2$. Then $Q^\theta(\mu)$ is unimodal with maximum achieved at $\hat{\mu}_Q^\theta < 1/2$ (Hopkins and Kornienko, 2007, Proposition 2). Moreover, $\lim_{\mu\to 0} Q^1(\mu) = \lim_{\mu\to 0} Q^0(\mu) = 1$ and

$$
\lim_{\mu\to 1} Q^1(\mu) = \lim_{\mu\to 1}\frac{1-F^1(\mu)}{1-\hat{F}^1(\mu)} = \lim_{\mu\to 1}\frac{f^1(\mu)}{\hat{f}^1(\mu)} = \lim_{\mu\to 1}\frac{f^0(\mu)}{\hat{f}^0(\mu)} = \lim_{\mu\to 1}\frac{1-F^0(\mu)}{1-\hat{F}^0(\mu)} = \lim_{\mu\to 1} Q^0(\mu).
$$

The proof concerns comparing the derivatives of Q^1 and Q^0, which are given by

$$
\frac{\mathrm{d}Q^1}{\mathrm{d}\mu} = \frac{f^1(\mu)}{1-\hat{F}^1(\mu)}\left(Q^1(\mu) - \frac{f^1(\mu)}{\hat{f}^1(\mu)}\right) \text{ and } \frac{\mathrm{d}Q^0}{\mathrm{d}\mu} = \frac{f^0(\mu)}{1-\hat{F}^0(\mu)}\left(Q^0(\mu) - \frac{f^0(\mu)}{\hat{f}^0(\mu)}\right).
$$

By MLRP and (OA.1), $f^1(\mu)/(1-\hat{F}^1(\mu)) < f^0(\mu)/(1-\hat{F}^0(\mu))$ for all μ.

Consider $\mu \geq \max\{\hat{\mu}_Q^1, \hat{\mu}_Q^0\}$, then both $Q^1(\mu)$ and $Q^0(\mu)$ are decreasing. Suppose there exists $\tilde{\mu}$ such that $Q^0(\tilde{\mu}) \leq Q^1(\tilde{\mu})$. Then at $\tilde{\mu}$, $\mathrm{d}Q^0/\mathrm{d}\mu < \mathrm{d}Q^1/\mathrm{d}\mu < 0$. This is a contradiction because $\lim_{\mu\to 1} Q^1(\mu) = \lim_{\mu\to 1} Q^0(\mu)$.

At $\mu = \max\{\hat{\mu}_Q^1, \hat{\mu}_Q^0\}$, one of $\mathrm{d}Q^1/\mathrm{d}\mu$ and $\mathrm{d}Q^0/\mathrm{d}\mu$ is zero and the other is strictly negative. As is shown above, $Q^1(\mu) < Q^0(\mu)$, so it must be that $\mathrm{d}Q^1/\mathrm{d}\mu < 0$ and $\mathrm{d}Q^0/\mathrm{d}\mu = 0$. This implies $\hat{\mu}_Q^1 < \hat{\mu}_Q^0$.

Consider $\mu \in (\hat{\mu}_Q^1, \hat{\mu}_Q^0)$, then Q^1 is decreasing and Q^0 is increasing. $\mathrm{d}Q^1/\mathrm{d}\mu < 0$ and $\mathrm{d}Q^0/\mathrm{d}\mu > 0$ implies $Q^1(\mu) < f^1(\mu)/\hat{f}^1(\mu) = f^0(\mu)/\hat{f}^0(\mu) < Q^0(\mu)$.

Consider $\mu \leq \hat{\mu}_Q^1$, then both $Q^1(\mu)$ and $Q^0(\mu)$ are increasing. Suppose there exists $\tilde{\mu}$ such that $Q^0(\tilde{\mu}) \leq Q^1(\tilde{\mu})$. Then at $\tilde{\mu}$, $0 < \mathrm{d}Q^1/\mathrm{d}\mu < \mathrm{d}Q^0/\mathrm{d}\mu$. This is a contradiction because $\lim_{\mu\to 0} Q^1(\mu) = \lim_{\mu\to 0} Q^0(\mu)$.

OA.2.2 Proof of Lemma 11

Let $h^\theta(\mu) = f^\theta(\mu)/(1 - F^\theta(\mu))$ denote the hazard rate conditional on θ. The posterior distribution conditional on $\theta = 0$ satisfies the definition of the ULR order: $F^0(\mu) \succ_{\mathrm{ULR}} \hat{F}^0(\mu)$. Then $h^0(\mu) > \hat{h}^0(\mu)$ for $\mu \geq 1/2$ (Hopkins and Kornienko, 2007, Corollary 1). The ULR order implies the ex ante distribution $\hat{F}$ is a mean-preserving spread of F (Hopkins and Kornienko, 2007, Proposition 1), so $F^1(\mu) + F^0(\mu) > \hat{F}^1(\mu) + \hat{F}^0(\mu)$ for $\mu \geq 1/2$. It then follows from Lemma 10 that $F^1(\mu) > \hat{F}^1(\mu)$.

OA.2.3 Proof of Lemma 12

For any two distributions $F \succ_{\mathrm{ULR}} \hat{F}$, $f/\hat{f}$ is unimodal. The likelihood ratio of F and $(1-\lambda)F + \lambda\hat{F}$ is $f/((1-\lambda)f + \lambda\hat{f})$ and the likelihood ratio of $(1-\lambda)F + \lambda\hat{F}$ and $\hat{F}$ is $((1-\lambda)f + \lambda\hat{f})/\hat{f}$. Both are unimodal as implied by that $f/\hat{f}$ is unimodal.

$F \succ_{\mathrm{ULR}} \hat{F}$ implies the mean of F is (weakly) higher than the mean of $\hat{F}$. So the mean of F is (weakly) higher than the mean of $(1-\lambda)F + \lambda\hat{F}$, which is (weakly) greater than the mean of $\hat{F}$. The result follows.

OA.2.4 Proof of Claim 4

The proof is mostly algebraic. For conciseness, I omit the argument of the functions. After some rearranging, $\mathcal{V}$ can be written in terms of h,

$$\mathcal{V} = \underbrace{q\left(1-\frac{1-\mu}{\mu}\right)}_{=:b} \underbrace{-q\left(1-\frac{1-\mu}{\mu}\right)\left(\frac{1-\mu}{\mu}\frac{1-F^1}{F^1}\right)}_{=:a} h.$$

That is, $\mathcal{V} = ah + b$. Let the superscript denote the (partial) derivative. Then $h^\lambda/h^\mu - \mathcal{V}^\lambda/\mathcal{V}^\mu = (h^\lambda/h^\mu)(a^\mu h + b^\mu)/\mathcal{V}^\mu - (a^\lambda h + b^\lambda)/\mathcal{V}^\mu$. Because $\mathcal{V}^\mu > 0$, $a^\mu h + b^\mu > -ah^\mu > 0$, showing Claim 4 is equivalent to showing $h^\lambda/h^\mu < (a^\lambda h + b^\lambda)/(a^\mu h + b^\mu)$. I prove the following chain of inequality: for all $\mu \geq 1/2$, $h^\lambda/h^\mu < q^\lambda/q^\mu < (a^\lambda h + b^\lambda)/(a^\mu h + b^\mu)$.

For the first inequality $h^\lambda/h^\mu < q^\lambda/q^\mu$, let $q = 1/(1 + m + dh)$ where

$$q = 1\Bigg/\left(1 + \underbrace{\frac{1-\mu}{\mu}\frac{1}{F^1}}_{=:m} \underbrace{-\left(\frac{1-\mu}{\mu}\right)^2 \frac{1-F^1}{F^1}}_{=:d} h\right).$$

It reduces to showing $h^\lambda/h^\mu - q^\lambda/q^\mu = (h^\lambda/h^\mu)\left(1 - h^\mu d/q^\mu\right) - (m^\lambda + d^\lambda h)/q^\mu < 0$. $h^\lambda < 0$ (Lemma 10), $h^\mu > 0$, $q^\mu > 0$, and $d < 0$, so $(h^\lambda/h^\mu)\left(1 - h^\mu d/q^\mu\right) < 0$. Note that $d = -m(1-\mu)/\mu + ((1-\mu)/\mu)^2$. Because $(1-F^0)/(1-F^1) < 1$ (MLRP) and $m^\lambda > 0$ (Lemma 11), $d^\lambda h = -m^\lambda(1-F^0)/(1-F^1) > -m^\lambda$, so $(m^\lambda + d^\lambda h)/q^\mu > 0$.

For the second inequality $q^\lambda/q^\mu < (a^\lambda h + b^\lambda)/(a^\mu h + b^\mu)$, the right-hand side is

$$\frac{q^\lambda \overbrace{\left(2-\frac{1}{\mu}\right)\left(1-\frac{1-F^0}{F^1}\right)}^{=:\alpha} \overbrace{-\left(\frac{1-F^1}{F^1}\right)^\lambda \frac{1-\mu}{\mu} bh}^{=:\beta}}{q^\mu \underbrace{\left(2-\frac{1}{\mu}\right)\left(1-\frac{1-F^0}{F^1}\right)}_{=\alpha} + \underbrace{\left(2-\frac{1}{\mu}\right)^\mu q\left(1-\frac{1-F^0}{F^1}\right) - \left(\frac{1-\mu}{\mu}\frac{1-F^1}{F^1}\right)^\mu bh}_{=:\eta}}.$$

It reduces to showing $q^\lambda/q^\mu - (a^\lambda h + b^\lambda)/(a^\mu h + b^\mu) = (q^\lambda/q^\mu)\eta/(q^\mu\alpha + \eta) - \beta/(q^\mu\alpha + \eta) < 0$. Because $q^\mu\alpha + \eta > 0$, it is equivalent to $q^\mu/q^\lambda - \eta/\beta > 0$. Writing out all the terms, this inequality follows from Lemma 10, Lemma 11, MLRP, IHRP, and symmetry.

OA.3 Omitted Proofs for Section 5

OA.3.1 Proof of Theorem 2

Equilibrium conditions

Leader-follower continuation game. Introducing a flow cost for the leader does not affect the follower's incentive. Same as the no-flow-cost case, the follower's first-order condition implies $x'(t) = \phi(x(t), y(t))$, where

$$\phi(x, y) := -r \left(\frac{\rho_0 f^1(y)(1 - F^1(x))(H - c) - (1 - \rho_0) f^0(y)(1 - F^0(x))(L + c)}{\rho_0 f^1(y) f^1(x)(H - c) - (1 - \rho_0) f^0(y) f^0(x)(L + c)} \right).$$

For leader of type x, same as before, denote his belief at the beginning of the leader-follower continuation game by $q_L(x) = \Pr(\theta = 1 | x, s_F < y(0))$. His expected payoff from disinvesting at t is

$$\begin{aligned}\mathcal{L}(x, t) =& q_L(x) \\ & \cdot \left(\int_0^t -y'(\tau) \frac{f^1(y(\tau))}{F^1(y(0))} \left(e^{-r\tau} H - \int_0^\tau e^{-r\tilde{\tau}} \eta \mathrm{d}\tilde{\tau} \right) \mathrm{d}\tau - \frac{F^1(y(t))}{F^1(y(0))} \int_0^t e^{-r\tilde{\tau}} \eta \mathrm{d}\tilde{\tau} \right) \\ & - (1 - q_L(x)) \\ & \cdot \left(\int_0^t -y'(\tau) \frac{f^0(y(\tau))}{F^0(y(0))} \left(e^{-r\tau} L + \int_0^\tau e^{-r\tilde{\tau}} \eta \mathrm{d}\tilde{\tau} \right) \mathrm{d}\tau + \frac{F^0(y(t))}{F^0(y(0))} \int_0^t e^{-r\tilde{\tau}} \eta \mathrm{d}\tilde{\tau} \right).\end{aligned}$$

The first-order condition implies $y'(t) = \psi(x(t), y(t))$, where

$$\psi(x, y) := -\eta \left(\frac{\rho_0 f^1(x) F^1(y) + (1 - \rho_0) f^0(x) F^0(y)}{\rho_0 f^1(x) f^1(y) H - (1 - \rho_0) f^0(x) f^0(y) L} \right).$$

Initial conditions. With strictly monotonic strategies, the flow cost does not affect the initial conditions. So the same as the no-flow cost case, $y(0) < z = x(0)$ and z's indifference condition implies $W_0(x(0), y(0)) = c$, where

$$W_0(x, y) := \frac{\rho_0 f^1(x)(F^1(x) - F^1(y)) H}{\rho_0 f^1(x) F^1(x) + (1 - \rho_0) f^0(x) F^0(x)} - \frac{(1 - \rho_0) f^0(x)(F^0(x) - F^0(y)) L}{\rho_0 f^1(x) F^1(x) + (1 - \rho_0) f^0(x) F^0(x)}.$$

Optimality

To show optimality, one needs to show (i) $\mathcal{F}(y, t)$ is single-peaked in t, (ii) $\mathcal{L}(x, t)$ is single-peaked in t, and (iii) all types above z invest and all types below do not. (i) is the same as the no-flow-cost case. The following lemma establishes (ii) holds. Given (i) and (ii), the proof of (iii) is the same as the no-flow-cost case.

Lemma OA.1. *For a fixed x, $\mathcal{L}(x,t)$ is single-peaked in t.*

Proof. The proof is analogous to the proof of Lemma 7. To simplify notation, define

$$M(x,t) := \frac{q_L(x)}{F^1(y(0))}(-y'(t))f^1(y(t))H - \frac{1-q_L(x)}{F^0(y(0))}(-y'(t))f^0(y(t))L,$$
$$N(x,t) := \left(\frac{q_L(x)}{F^1(y(0))}F^1(y(t)) + \frac{1-q_L(x)}{F^0(y(0))}F^0(y(t))\right)\eta.$$

In words, $e^{-rt}M(x,t)\mathrm{d}t$ is type x's marginal benefit from waiting for $\mathrm{d}t$ before disinvesting and $e^{-rt}N(x,t)\mathrm{d}t$ is the marginal cost. Let the subscript i denote the partial derivative with respect to the i-th argument. The first-order condition of $\mathcal{L}$ implies $M(x(t),t) = N(x(t),t)$. Because strategies are strictly monotone and everywhere differentiable, at each t, there exists one and only one type whose first-order condition is satisfied at t. Denote the type whose first-order condition is satisfied at t^* by x^*, that is, $M(x^*,t^*) = N(x^*,t^*)$. Suppose x^* mimics the behavior of type $\hat{x}$ by stopping at $\hat{t}$. Because $M(x,t)$ is differentiable in x, by the fundamental theorem of calculus,

$$M(x^*,\hat{t}) = M(\hat{x},\hat{t}) + \int_{\hat{x}}^{x^*} M_1(x,\hat{t})\mathrm{d}x = N(\hat{x},\hat{t}) + \int_{\hat{x}}^{x^*} M_1(x,\hat{t})\mathrm{d}x,$$

where $M_1(x,\hat{t}) = \mathrm{d}M(x,\hat{t})/\mathrm{d}x$. The second equality follows from $\hat{x}$'s first-order condition $M(\hat{x},\hat{t}) = N(\hat{x},\hat{t})$. By MLRP, $q_L(x)$ is decreasing in x and because $y'(t) < 0$, so $M_1(x,\hat{t}) > 0$. Thus, if $\hat{x} < x^*$, then

$$M(x^*,\hat{t}) = N(\hat{x},\hat{t}) + \int_{\hat{x}}^{x^*} M_1(x,\hat{t})\mathrm{d}x > N(\hat{x},\hat{t}) > N(x^*,\hat{t}),$$

where the first inequality follows from $\int_{\hat{x}}^{x^*} M_1(x,\hat{t})\mathrm{d}x > 0$, and the second inequality follows from that N is decreasing in x because of MLRP and $y(t) < y(0)$. Similarly, if $\hat{x} > x^*$, then $\int_{\hat{x}}^{x^*} M_1(x,\hat{t})\mathrm{d}x < 0$, so

$$M(x^*,\hat{t}) = N(\hat{x},\hat{t}) + \int_{\hat{x}}^{x^*} M_1(x,\hat{t})\mathrm{d}x < N(\hat{x},\hat{t}) < N(x^*,\hat{t}).$$

$x(t)$ is increasing, so $\hat{x} < (>)x^*$ is equivalent to $\hat{t} < (>)t^*$. The above argument shows

$M(x^*, \hat{t}) - N(x^*, \hat{t}) > 0$ for all $\hat{t} < t^*$ and $M(x^*, \hat{t}) - N(x^*, \hat{t}) < 0$ for all $\hat{t} > t^*$. $\square$

Existence

In any dynamic equilibrium in strictly monotonic and differentiable strategies,
(i) by optimality, players must get strictly positive payoff;
(ii) strategies are strictly monotone: $x'(t) > 0$ and $y'(t) < 0$ for all $t \geq 0$;
(iii) strategies are differentiable for all $t \geq 0$ and $x(t), y(t) \in (0, 1)$.

(i) In the leader-follower game, for the leader, disinvesting at $t = 0$ generates payoff 0 for any types of the leader, that is, $\mathcal{L}(x, 0) = 0$ for all $x \geq x(0)$. By Lemma OA.1, $\mathcal{L}(x, t)$ is single-peaked in t, so by optimality, if a type optimally disinvests at $t > 0$, he must expect to get a strictly higher payoff than disinvesting at $t = 0$. That is, $\mathcal{L}(x(t), t) > \mathcal{L}(x(t), 0) = 0$ for all $x(t) > x(0)$. For the follower, $\mathcal{F}(y(t), t) > 0$ if and only if

$$\frac{\rho_0}{1-\rho_0} \frac{f^1(y(t))}{f^0(y(t))} \frac{1 - F^1(x(t))}{1 - F^0(x(t))} > \frac{L + c}{H - c}. \tag{OA.2}$$

I now show players' expected payoff at the beginning of the game is positive. Note that

$$\frac{\rho_0}{1-\rho_0} \frac{f^1(x(0))}{f^0(x(0))} \frac{1 - F^1(x(0))}{1 - F^0(x(0))} > \frac{\rho_0}{1-\rho_0} \frac{f^1(y(0))}{f^0(y(0))} \frac{1 - F^1(x(0))}{1 - F^0(x(0))} > \frac{L + c}{H - c},$$

where the first inequality follows from $x(0) > y(0)$, and the second inequality follows from evaluating (OA.2) at $t = 0$. This implies z's ex ante expected payoff is strictly positive. By MLRP, all types above z receive strictly positive payoffs. Types below z do not invest at the beginning of the game so their payoff is at least 0.

(ii) $y'(t) < 0$ if and only if

$$\frac{\rho_0}{1-\rho_0} \frac{f^1(y(t))}{f^0(y(t))} \frac{f^1(x(t))}{f^0(x(t))} > \frac{L}{H}. \tag{OA.3}$$

Given (OA.2), $x'(t) > 0$ if and only if

$$\frac{\rho_0}{1-\rho_0} \frac{f^1(y(t))}{f^0(y(t))} \frac{f^1(x(t))}{f^0(x(t))} < \frac{L + c}{H - c}. \tag{OA.4}$$

(iii) Because $\phi(\cdot, \cdot)$ and $\psi(\cdot, \cdot)$ are autonomous first-order differential equations and are continuous for all (x, y) such that $\phi(x, y) > 0$ and $\psi(x, y) < 0$, and $x(t)$ and $y(t)$ are bounded, so as $t \to \infty$, $x'(t) \to 0$ and $y'(t) \to 0$. Note that $x'(t) = 0$ and $y'(t) = 0$

if and only if $x(t) = 1$ and $y(t) = 0$. So $\phi(x(t), y(t)) \to 0$ and $\psi(x(t), y(t)) \to 0$ if and only if $x(t) \to 1$ and $y(t) \to 0$.

Define $\mathcal{D} \subset (0,1)^2$ and $\mathcal{D}_0 \subset (0,1)^2$ as

$$\mathcal{D} := \{(x,y) : \text{(OA.2)}, \text{(OA.3)} \text{ and } \text{(OA.4)} \text{ hold}\},$$

$$\mathcal{D}_0 := \mathcal{D} \cap \{(x,y) : x > y \text{ and } V(x,y) = c\}.$$

In words, if a solution $(x(t), y(t))$ to the differential system (9) is an equilibrium, then it must be that $(x(t), y(t)) \in \mathcal{D}$ for all $t \geq 0$ with initial values $(x(0), y(0)) \in \mathcal{D}_0$.

It is helpful to consider the (x,y)-plane and the differential equation

$$y'(x) = \Upsilon(x,y) := \frac{\psi(x,y)}{\phi(x,y)}, \; \forall (x,y) \in \mathcal{D}. \tag{OA.5}$$

By definition, $\Upsilon(x,y)$ is continuous in (x,y) for all $(x,y) \in \mathcal{D}$. An equilibrium is a solution $y(x)$ to the differential equation (OA.5) in $\mathcal{D}$ with $y(x) < x$ that goes through a point in $\mathcal{D}_0$ and converges to 0 as x goes to 1. Showing an equilibrium exists and is unique is equivalent to showing such solution exists and is unique. In what follows, Lemma OA.2 shows there exists a trajectory in $\mathcal{D}$ that converges to 0 as x goes to 1. Under parametric restriction (OA.12), this trajectory is unique. Lemma OA.3 shows this (unique) trajectory goes through one and only one point in $\mathcal{D}_0$ for $y(x) < x$. Thus the equilibrium is unique.

Figure OA.1 illustrates the unique equilibrium trajectory (red arrowed curve) which goes through exactly one point in $\mathcal{D}_0$ and converges to the point $(1,0)$. All other trajectories (black arrowed curves) will diverge to the boundaries of $\mathcal{D}$. Figure OA.1 also displays annotations that facilitate the rest of the proof.

Lemma OA.2. *For any feasible parameters, there exists a solution $y(x)$ to the differential equation (OA.5) in $\mathcal{D}$ with $y(x) \to 0$ as $x \to 1$.*

Proof. Consider the boundaries of $\mathcal{D}$. For any fixed $x \in (0,1)$, let $\beta_F(x)$ be such that

$$\frac{\rho_0}{1-\rho_0} \frac{f^1(\beta_F(x))}{f^0(\beta_F(x))} \frac{1-F^1(x)}{1-F^0(x)} = \frac{L+c}{H-c}, \tag{OA.6}$$

$\beta_f(x)$ be such that

$$\frac{\rho_0}{1-\rho_0} \frac{f^1(\beta_f(x))}{f^0(\beta_f(x))} \frac{f^1(x)}{f^0(x)} = \frac{L}{H}, \tag{OA.7}$$

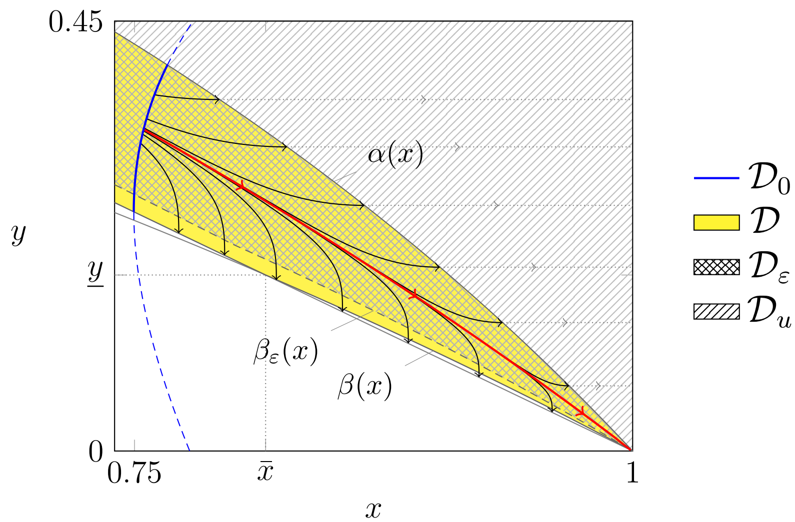

Figure OA.1: Equilibrium trajectory (red arrowed curve) and sample trajectories (non-equilibrium, black arrowed curves) to the differential system (9) for $\rho_0 = 1/2, H = L = 1, r = 1/5, c = 0.38, \eta = 1/20$ and posterior beliefs distributed according to $Beta(1+\theta, 1+(1-\theta))$.

and $\alpha(x)$ be such that

$$\frac{\rho_0}{1-\rho_0}\frac{f^1(\alpha(x))}{f^0(\alpha(x))}\frac{f^1(x)}{f^0(x)} = \frac{L+c}{H-c}. \tag{OA.8}$$

Finally, define

$$\beta(x) := \max_{x\in(0,1)} \{\beta_F(x), \beta_f(x)\}.$$

By IHRP, $\beta_f(x)$ and $\beta_F(x)$ intersect at most once for $x \in (0,1)$.

Claim OA.1. (i) $\mathcal{D}$ is non-empty. (ii) $(1,0) \in \mathrm{cl}(\mathcal{D})$ and $(0,1) \in \mathrm{cl}(\mathcal{D})$.

Proof. (i) Fix $x \in (0,1)$. By MLRP, the left-hand side of (OA.6) evaluated at any $(x', y') > (x, \beta_F(x))$ is strictly higher than $(L+c)/(H-c)$, the left-hand side of (OA.7) evaluated at any $(x', y') > (x, \beta_f(x))$ is strictly higher than L/H, and the left-hand side of (OA.8) evaluated at any $(x', y') < (x, \alpha(x))$ is strictly lower than $(L+c)/(H-c)$. $\alpha(x) > \beta(x)$ for all $x \in (0,1)$. So $\mathcal{D}$ is non-empty.

(ii) Fix $x \in (0,1)$. Consider (OA.6). Take the limit of both sides as $x \to 1$. The right-hand side is constant at $(L+c)/(H-c)$. On the left-hand side, because $\lim_{x\to 1} \frac{1-F^1(x)}{1-F^0(x)} = \lim_{x\to 1} \frac{f^1(x)}{f^0(x)} = \infty$,it must be $f^1(\beta_F(x))/f^0(\beta_F(x)) \to 0$, which means $\beta_F(x) \to 0$. The same argument applies for equations (OA.7) and (OA.8). This implies $(1,0) \in \mathrm{cl}(\mathcal{D})$. An analogous argument shows $(0,1) \in \mathrm{cl}(\mathcal{D})$. $\square$

By definition, for all $(x, y) \in \mathcal{D}$, $\psi(x, y) < 0$ and $\phi(x, y) > 0$, so $\Upsilon(x, y) < 0$. Define $\mathcal{D}_u \in (0, 1)^2$ (the subscript u stands for "upper") as

$$\mathcal{D}_u := \left\{ (x, y) : \frac{\rho_0}{1 - \rho_0} \frac{f^1(y)}{f^0(y)} \frac{f^1(x)}{f^0(x)} \geq \frac{L + c}{H - c} \right\}.$$

In words, $\mathcal{D}_u$ is the set of points in the (x, y)-plane that are equal to or above $\alpha(x)$. By definition and the continuity of the distribution functions, $\mathcal{D} \cup \mathcal{D}_u$ is connected. For any fixed $x \in (0, 1)$, as $y \to \alpha(x)$, $\Upsilon(x, y) \to 0$. Let $\Upsilon(x, y) = 0$ for all $(x, y) \in \mathcal{D}_u$. Then $\Upsilon(x, y)$ is continuous in (x, y) for all $(x, y) \in \mathcal{D} \cup \mathcal{D}_u$. Apply the implicit function theorem to (OA.8), MLRP implies for all feasible parameters and $x \in (0, 1)$,

$$\alpha'(x) < 0 = \Upsilon(x, \alpha(x)).$$

This means $\alpha(x)$ is a strong lower fence (or lower solution, see Hubbard and West, 1991, Section 1.3, or Teschl, 2012, Section 1.5) for the differential equation

$$y'(x) = \Upsilon(x, y) = \begin{cases} \psi(x, y)/\phi(x, y) & (x, y) \in \mathcal{D} \\ 0 & (x, y) \in \mathcal{D}_u \end{cases}. \tag{OA.9}$$

Consider an ε-variation of $\beta_F(x)$ and $\beta_f(x)$. Let $\beta_{F,\varepsilon}(x)$ be such that

$$\frac{\rho_0}{1 - \rho_0} \frac{f^1(\beta_{F,\varepsilon}(x))}{f^0(\beta_{F,\varepsilon}(x))} \frac{1 - F^1(x)}{1 - F^0(x)} = \frac{L + c}{H - c} + \varepsilon, \tag{OA.10}$$

and $\beta_{f,\varepsilon}(x)$ be such that

$$\frac{\rho_0}{1 - \rho_0} \frac{f^1(\beta_{f,\varepsilon}(x))}{f^0(\beta_{f,\varepsilon}(x))} \frac{f^1(x)}{f^0(x)} = \frac{L}{H} + \varepsilon. \tag{OA.11}$$

Define

$$\beta_\varepsilon(x) := \max_{x \in (0,1)} \{\beta_{F,\varepsilon}(x), \beta_{f,\varepsilon}(x)\},$$

$$\mathcal{D}_\varepsilon := \{(x, y) : x \in (0, 1) \text{ and } \beta_\varepsilon(x) \leq y < \alpha(x)\}.$$

By MLRP, for all $x \in (0, 1)$, $\beta_{F,\varepsilon}(x) < \alpha(x)$. For all $\varepsilon < (L + c)/(H - c) - L/H$, $\beta_{f,\varepsilon}(x) < \alpha(x)$. By the same argument as Claim OA.1, $\mathcal{D}_\varepsilon$ is non-empty, and the points $(1, 0)$ and $(0, 1)$ are in the closure of $\mathcal{D}_\varepsilon$. Moreover, $\mathcal{D}_\varepsilon \cup \mathcal{D}_u$ is connected and $\Upsilon(x, y)$ is continuous in (x, y) for all $(x, y) \in \mathcal{D}_\varepsilon \cup \mathcal{D}_u$.

Apply the implicit function theorem to (OA.10) and (OA.11), MLRP implies that for all feasible parameters and any $\varepsilon > 0$, $\beta'_{F,\varepsilon}(x)$ and $\beta'_{f,\varepsilon}(x)$ are both finite and negative. Therefore $\beta'_{\varepsilon}(x) > -\infty$ for all $x \in (0, 1)$.

Claim OA.2. There exists $\hat{\varepsilon} > 0$ such that $\Upsilon(x, \beta_{\hat{\varepsilon}}(x)) < \beta'_{\hat{\varepsilon}}(x)$ for all x.

Proof. For all $x \in (0, 1)$, by definition, as $\varepsilon \to 0$, $\beta_{\varepsilon}(x) \to \beta(x)$, which implies $\Upsilon(x, \beta_{\varepsilon}(x)) \to -\infty$. So for any x, there exists $\varepsilon(x) > 0$ (ε might depend on x) such that for all $\varepsilon < \varepsilon(x)$, $\Upsilon(x, \beta_{\varepsilon}(x)) < \beta'_{\varepsilon}(x)$. Let $\hat{\varepsilon} := \inf_{x \in (0,1)} \varepsilon(x)$. It remains to show $\hat{\varepsilon} > 0$. Suppose $\hat{\varepsilon} = 0$. Then there exists a sequence ε_n with $\varepsilon_n \to 0$ such that for each ε_n there exists x_n such that $\Upsilon(x_n, \beta_{\varepsilon_n}(x_n)) \geq \beta'(x_n)$. This is a contradiction because for all x_n, $\beta'(x_n) > -\infty$ but as $\varepsilon_n \to 0$, $\Upsilon(x_n, \beta_{\varepsilon_n}(x_n)) \to -\infty$. □

This means $\beta_{\varepsilon}(x)$ is a strong upper fence (or upper solution) for the differential equation (OA.9). Therefore, in $\mathcal{D}_{\varepsilon} \cup \mathcal{D}_u$, there exists a solution $y(x)$ to the differential equation (OA.5) with $\beta_{\varepsilon}(x) \leq y(x) \leq \alpha(x)$ for all $x \in (0, 1)$ (see Hubbard and West, 1991, Theorem 1.4.4, or Teschl, 2012, Lemma 1.2).

The above argument establishes there exists a solution in $\mathcal{D}_{\varepsilon} \cup \mathcal{D}_u$. It remains to show that the solution is within $\mathcal{D}_{\varepsilon}$ (and thus within $\mathcal{D}$), not in $\mathcal{D}_u$. This boils down to showing that solutions in $\mathcal{D}_u$ do not converge to 0 as $x \to 1$. This follows from the definition that $y'(x) = 0$ for all $(x, y) \in \mathcal{D}_u$. So for any $(x, y(x)) \in \mathcal{D}_u$ that solves the differential equation (OA.9), $y(x) > 0$ for all x. □

Uniqueness

Assumption. *Assume the following condition holds:*

$$\forall (x, y) \in \mathcal{D},\ \partial\Upsilon(x, y)/\partial y \geq 0. \tag{OA.12}$$

The uniqueness of a global condition can be established if the primitives satisfy the above condition. It can be numerically verified that (OA.12) is satisfied if f^{θ} is induced by signals distributed according to the Beta distributions or the Normal distributions. Moreover, by definition, as $x \to 1$, $\alpha(x) \to 0$ and $\beta_{\varepsilon}(x) \to 0$, so

$$\lim_{x \to 1} |\alpha(x) - \beta_{\varepsilon}(x)| = 0. \tag{OA.13}$$

Conditions (OA.12) and (OA.13) imply the solution is unique in $\mathcal{D}_{\varepsilon}$ (see Hubbard and West, 1991, Theorem 1.4.5, or Teschl, 2012, Section 1.5).

The above argument establishes the unique solution is in $\mathcal{D}_\varepsilon$. It remains to show this solution is unique in $\mathcal{D}$. Because $\mathcal{D} = \mathcal{D}_\varepsilon \cup \{(x, y) : x \in (0, 1) \text{ and } \beta(x) < y < \beta_\varepsilon(x)\}$, it boils down to showing there does not exist a solution in the set $\{(x, y) : x \in (0, 1) \text{ and } \beta(x) < y < \beta_\varepsilon(x)\}$. For all $y(x)$ such that $\beta(x) < y(x) < \beta_\varepsilon(x)$, $y'(x) \to -\infty$, which implies for all $x \in (0, 1)$, $y(x) \to \beta(x) > 0$.

Denote this unique solution by $\hat{y}(x)$. I prove there exists a unique set of initial values satisfying $\hat{y}(x)$. This is summarized in the following lemma.

Lemma OA.3. *There exists a unique* $(x_0, y_0) \in \mathcal{D}_0$ *such that* $y_0 = \hat{y}(x_0)$.

Proof. To simplify notation, define

$$\ell(x, y) := \frac{\rho_0}{1-\rho_0} \frac{f^1(y)}{f^0(y)} \frac{f^1(x)}{f^0(x)}.$$

Recall that $\mathcal{D}_0$ is the set of points $(x, y) \in \mathcal{D}$ that satisfies the equation $W_0(x, y) = c$. Solve $W_0(x, y) = c$ for y in terms of x and denote the solution by $y_{W_0}(x)$. By Claim 6 (iii) and (iv), $y_{W_0}(x)$ is increasing and continuous in x for all x such that $y_{W_0}(x) < x$.

By a change of variable, Lemma OA.2 shows $\hat{y}(x)$ also converges to 1 as $x \to 0$. So $\hat{y}(x)$ is a strictly decreasing function that converges to 1 as $x \to 0$ and converges to 0 as $x \to 1$, and satisfies $\ell(x, \hat{y}(x)) \in (L/H, (L + c)/(H - c))$ for all $x \in (0, 1)$. So points in $\mathcal{D}_0$ constitute a strictly increasing and continuous function that starts at a point below $\hat{y}(x)$, and ends at a point above $\hat{y}(x)$. The result follows. $\square$

References

Hopkins, Ed and Tatiana Kornienko (2007) "Cross and Double Cross: Comparative Statics in First Price and All Pay Auctions," *The B.E. Journal of Theoretical Economics*, 7.

Hubbard, John H. and Beverly H. West (1991) *Differential Equations: A Dynamical Systems Approach*, 5 of Texts in Applied Mathematics, New York, NY: Springer New York.

Teschl, Gerald (2012) *Ordinary Differential Equations and Dynamical Systems*, Providence, R.I: American Mathematical Society.